Byzantine Vector Consensus in Complete Graphs

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Byzantine Vector Consensus Model

Configurations:

1. fully connected network.
2. n processes, with at most f Byzantine processes.
3. d-dimensional real-valued vector as input.

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2. n processes, with at most f Byzantine processes.
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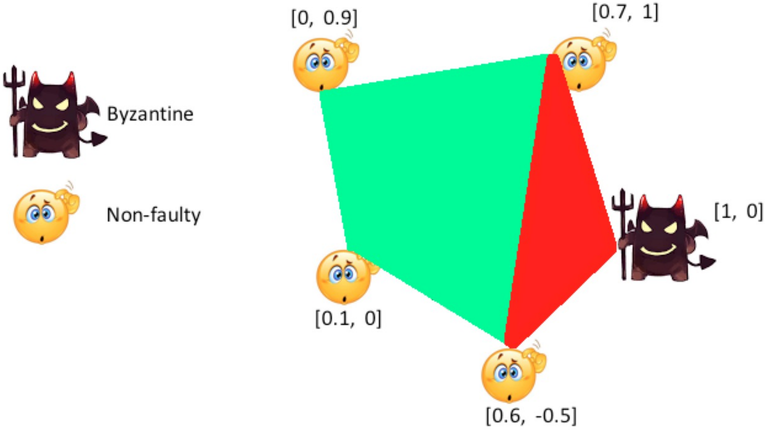
Conditions:

1. **Termination:** Each non-faulty process must terminate after a finite amount of time.
2. **Agreement:** The decision (or output) vector at all the non-faulty processes must be identical.
3. **Validity:** The decision vector at each non-faulty process must be in the **convex hull** of the input vectors at the non-faulty processes.

Byzantine Vector Consensus Model



An example with 4 non-faulty processes and 1 byzantine process.

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Can we simply perform Byzantine Agreement on each dimension of the input vectors independently?

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Counterexample:

We have 4 process, and only one is faulty.

p_0 : is faulty.

$p_1 : [1, 0, 0]$

$p_2 : [0, 1, 0]$

$p_3 : [0, 0, 1]$

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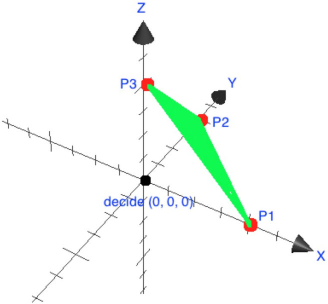

if we perform Byzantine Agreement on each dimension of the vectors separately, then the processes may possibly agree on $[0, 0, 0]$. In this case, the *Validity* condition is violated!

Results in the paper

In this paper, the authors obtain the following two results for BVC in *complete graph* while tolerating up to f Byzantine failures when the input is a d-dimensional vector:

1. For a synchronous system, $n > max(3f, (d+1)f)$ is necessary and sufficient for achieving BVC.

Results in the paper

In this paper, the authors obtain the following two results for BVC in *complete graph* while tolerating up to f Byzantine failures when the input is a d-dimensional vector:

1. For a synchronous system, $n > max(3f, (d+1)f)$ is necessary and sufficient for achieving BVC.
2. For an asynchronous system, $n > (d+2)f$ is necessary and sufficient to achieve *approximate* BVC.

Necessary condition

Theorem 1 *$n > max(3f, (d+1)f)$ is necessary for BVC in a synchronous system.*

Proof (Necessary)

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1. When $d = 1$, $n > 3f$ is a necessary condition for achieving Byzantine agreement in presence of up to f faults. (Already proved in the textbook!)

Proof (Necessary)

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1. When $d = 1$, $n > 3f$ is a necessary condition for achieving Byzantine agreement in presence of up to f faults. (Already proved in the textbook!)
2. When $d \geq 2$, $n > (d+1)f$ is also a necessary condition.

Proof (Necessary)

Theorem 1 $n > max(3f, (d+1)f)$ *is necessary for BVC in a synchronous system.*

Consider the validity condition: **decision vector should be in the convex hull of non-fault processes' inputs !**

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Claim: Suppose f $= 1$, since none of the non-faulty process know which process is faulty, the decision vector v must be in the convex hull of each multiset containing the input vectors of $n-1$ of the processes (there are n such multiset, let its convex hull be Q_i for $1 = 1, 2, ..., n$).

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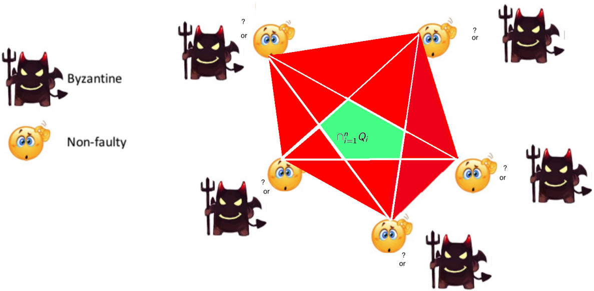
1. p_i is the input of process i
2. $\mathcal{P} = \{p_1, p_2, ..., p_n\}$
3. $H(\mathcal{P})$ is the convex hull of $\mathcal{P}$.
4. $Q_i = \mathcal{H}(\mathcal{P} - \{p_i\})$
5. $v \in \cap_{i=1}^{n} Q_i$

Proof (Necessary)



Proof (Necessary)

Theorem 1 *$n > max(3f, (d+1)f)$ is necessary for BVC in a synchronous system.*

Proof:

1. $f = 1$, input vector $p_i, 1 \leq i \leq d$, is a vector whose i-th element is 1 and the remaining are 0. p_{d+1} is the all-0 vector.

$$\left.\begin{array}{ll} p_1 = & [1 \quad 0 \quad \dots \quad 0] \\ p_2 = & [0 \quad 1 \quad \dots \quad 0] \\ \vdots & \qquad\qquad \vdots \\ p_d = & [0 \quad 0 \quad \dots \quad 1] \\ p_{d+1} = & \underbrace{[0 \quad 0 \quad \dots \quad 0]}_{\text{d dimensional}} \end{array}\right\} \text{d+1 input vectors}$$

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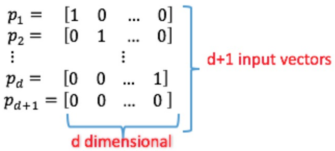

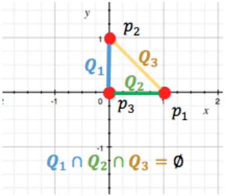

Let $\mathcal{P} = \{p_1, .., p_{d+1}\}$ and Q_i is the convex hull of $\mathcal{P} - \{p_i\}$, then $\cap_{i=1}^{d+1} Q_i = \emptyset$.

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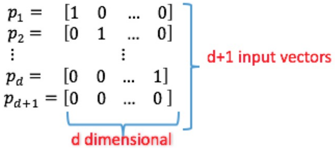

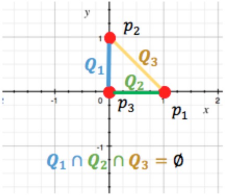

Let $\mathcal{P} = \{p_1, .., p_{d+1}\}$ and Q_i is the convex hull of $\mathcal{P} - \{p_i\}$, then $\cap_{i=1}^{d+1} Q_i = \emptyset$.
Which leads to $n \leq d+1$ is not sufficient. Therefore, $n \geq d+2$ is necessary for the case when $f = 1$.

Proof (Necessary)

Theorem 1 *$n > max(3f, (d+1)f)$ is necessary for BVC in a synchronous system.*

For the following two cases.

1. $f = 1$, the input vector for $p_i, 1 \leq i \leq d$, is a vector whose i-th element is 1 and the remaining are 0. p_{d+1} is the all-0 vector.
2. $f > 1$, we can use the simulation approach, and thus f simulated process are implemented by a single process.

Proof (Necessary)

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2. $f > 1$, we can use the simulation approach. That is, we use a single process to simulate f processes. Therefore, if a correct algorithm were to exist for tolerating up to f faults among $(d+1)f$ processes, then we can obtain a correct algorithm to tolerate a single failure among $d+1$ processes. Contradict to the case $f = 1$.

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Therefore, $n > max(3f, (d+1)f)$ is necessary for achieving BVC in a synchronous system.$\square$

Proof of Sufficient Condition

Theorem 2 *$n > max(3f, (d+1)f)$ is sufficient for achieving BVC in a synchronous system.*

Define:

- Y: a multiset of points. (e.g. $\mathrm{Y} = \{1, 1, 2, 3\}$)
- $\mathcal{H}(T)$: the convex hull of a multiset T.
- $\Gamma(Y) = \cap_{T \subseteq Y, |T| = |Y| - f} \mathcal{H}(T)$: intersection of convex hulls of all subsets of Y of size $|Y| - f$.

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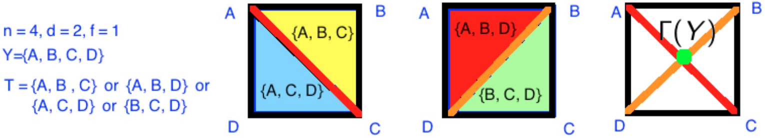

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Algorithm: ($n \geq max(3f+1, (d+1)f+1)$)

1. Each process use the Byzantine agreement algorithm to decide the d elements one by one of all the n processes. Non-faulty processes can agree on the d elements of the input vector at each of the n processes, and thus collect such n vectors as S.
2. Each process chooses as its decision vector a point in $\Gamma(S)$ using a determinstic algorithm.

Proof of Termination

- *Termination:* The Byzantine agreement algorithm terminates in finite time.
-
-

Proof of Agreement

- *Termination:* The Byzantine agreement algorithm terminates in finite time.
- *Agreement:* Agreement condition holds because all the non-faulty processes have identical multiset S, thus we can use a deterministic algorithm to pick the decision vector.

Proof of the correctness of the algorithm

- *Termination:* The Byzantine agreement algorithm terminates in finite time.
- *Agreement:* Agreement condition holds because all the non-faulty processes have identical multiset S, thus we can use a deterministic algorithm to pick the decision vector.
- *Validity:*

Proof of Validity

Theorem 3 (Tverberg's Theorem) *For an integer $f \geq 1$, and for every multiset Y containing at least $(d+1)f+1$ points in R^d, there exists a partition $Y_1, ..., Y_{f+1}$ of Y into $f+1$ non-empty multisets such that $\cap_{l=1}^{f+1} \mathcal{H}(Y_l) \neq \emptyset$*

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1. f: an integer and ≥ 1.
2. Y: a multiset, and $|Y| = (d+1)f+1$.
3. $\cap_{l=1}^{f+1} H(Y_l) \neq \emptyset$, for some partition $Y_1, ..., Y_{f+1}$ of Y. (Note: $\cup_{l=1}^{f+1} Y_l = Y$)

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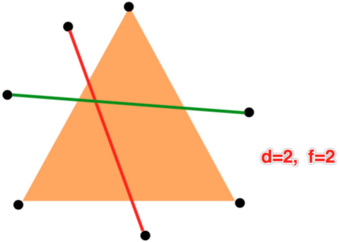

Figure 1: *Illustration of a Tverberg partition.*
Acknowledgment: *The above example is inspired by an illustration authored by David Eppstein, which is available in the public domain from* Wikipedia Commons.

Proof of Validity

Lemma 1 *For any multiset* Y *containing at least* $(d+1)f+1$ *points in* $\mathbb{R}^d$, $\Gamma(Y) \neq \emptyset$.
(Recall that: $\Gamma(Y) = \cap_{T \subseteq Y, |T|=|Y|-f} \mathcal{H}(T)$)

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Proof:

1. By Tverberg's theorem, there exists partition of Y ($|Y| \geq (d+1)f+1$) into non-empty subsets $Y_1, ..., Y_{f+1}$, such that $\cap_{l=1}^{f+1} \mathcal{H}(Y_l) \neq \emptyset$.

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2. Consider $\Gamma(Y) = \cap_{T \subseteq Y, |T|=|Y|-f} \mathcal{H}(T)$.
We have $|T| = |Y| - f$, and there are $f+1$ subsets in the Tverberg's partition of Y. Therefore, at least one subset Y_i is in T. Hence, $\cap_{l=1}^{f+1} \mathcal{H}(Y_l) \subseteq \Gamma(Y)$, and thus $\Gamma(Y) \neq \emptyset$. $\square$

Proof of Validity

- *Termination:* The Byzantine agreement algorithm terminates in finite time.
- *Agreement:* Agreement condition holds because all the non-faulty processes have identical multiset S, thus we can use a deterministic algorithm to pick the decision vector.
- *Validity:* With at most f faulty process, there at least one multiset T^* must contain the inputs of only non-faulty processes. Thus, $\Gamma(S)$ is in the convex hull of the inputs of non-faulty processes. Hence, validity is satisfied.

Conclusion

For a synchronous system, $n > max(3f, (d+1)f)$ is necessary and sufficient for achieving BVC.

Further Reading

See next slide for the Asynchronous case.

Background

How about in an asynchronous system?

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- In an asynchronous system, *exact* consensus is impossible in the presence of faulty processes.
- But we can prove that $n \geq (d+2)f + 1$ is necessary and sufficient to achieve *approximate* Byzantine vector consensus.

Proof (Necessary)

Theorem: $n \geq (d+2)f + 1$ is necessary for approximate BVC in an asynchronous system.

Necessary (Cnt.)

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We only need to consider the case when $f = 1$, and for the cases that $f \geq 2$, we can use a simulation similar to the proof of Theorem 1.

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Theorem: $n \geq (d+2)f + 1$ is necessary for approximate BVC in an asynchronous system.

Suppose that $f = 1$, and $n = d + 2$. We can see that the following d+1 scenarios cannot be ditinguished by processes $p_1, p_2, ..., p_{d+1}$.

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Theorem: $n \geq (d+2)f + 1$ is necessary for approximate BVC in an asynchronous system.

Suppose that $f = 1$, and $n = d + 2$. We can see that the following d+1 scenarios cannot be ditinguished by processes $p_1, p_2, ..., p_{d+1}$.

- Process p_{d+2} has crashed.
- Process p_j $(j \neq i, 1 \leq j \leq d+1)$ is faulty, and process p_{d+2} is slow.

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In order to meet the validity condition, the decided vector of p_i must be in the intersection of convex hull of all non-faulty processes' vectors. For the first case, it should be in the convex hull of X_i^{d+2}. For the other d cases, it should be in the convex hull of X_i^j. Where,

$$X_i^j = \{x_k : k \neq j \textit{ and } 1 \leq k \leq d+1\}$$

.

Besides, we have:

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so as to meet the validity condition. Consider the following example:

$$\left.\begin{array}{ll}
x_1 = & [4\epsilon \quad 0 \quad \dots \quad 0] \\
x_2 = & [0 \quad 4\epsilon \quad \dots \quad 0] \\
\vdots & \qquad \vdots \\
x_d = & [0 \quad 0 \quad \dots \quad 4\epsilon] \\
x_{d+1} = & [0 \quad 0 \quad \dots \quad 0] \\
x_{d+2} = & \underbrace{[0 \quad 0 \quad \dots \quad 0]}_{\text{d dimensional}}
\end{array}\right\} \text{d+2 input vectors}$$

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The decision vector of p_i must be x_i, and thus for each pair of processes in $p_1, \dots, p_{d+1}$ differ by 4ϵ in at least one element!!

Necessary (Cnt.)

So far, we have proved that $n \leq d+2$ is not sufficient, and for the case when $f > 1$, we can use a simulation similar to the proof of Theorem 1, and show that $n \leq (d+2)f$ is also not sufficient. Thus, $n \geq (d+2)f+1$ is necessary for $f \geq 1$.

In the following, we will present a prove that $n \geq (d+2)f+1$ is sufficient by proving the correctness of an algorithm.

Abraham, Amit and Dolev's (AAD) algorithm

Abraham, Amit and Dolev's (AAD) algorithm aims to solve the approximate *scalar* consensus problem in an asynchronous system. It could be viewed as consisting of three components:

- **AAD componend #1**: each process communicate its state vector $v_i[t-1]$ to other processes. AAD guarantees that for each non-faulty process p_i in round t obtains a set $B_i[t]$ containing **at least** $n-f$ tuples of the form (p_j, w_j, t) such that the following properties hold:
  - If p_i, p_j are non-faulty, then $|B_i[t] \cap B_j[t]| \geq n-f$

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  - If p_i, p_j are non-faulty, then $|B_i[t] \cap B_j[t]| \geq n-f$
  - If (p_l, w_l, t) and (p_k, w_k, t) are both in $B_i[t]$, then $p_l \neq p_k$.
  - If p_k is non-faulty, and $(p_k, w_k, t) \in B_i[t]$, then $w_k = v_k[t-1]$.

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- **AAD component #2**: Process p_i, having obtained $B_i[t]$, computes its new state $v_i[t]$ as a function of the tuples in $B_i[t]$.

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- **AAD component #3**: AAD also includes a sub-algorithm that allows the non-faulty processes to determine when to terminate their computation.

Asynchronous Approximate BVC algorithm ($n \geq (d+2)f+1$)

1. In the t-th round, each non-faulty process uses the mechanism in *Component #1* of the AAD algorithm to obtain a set $B_i[t]$ containing at least $n-f$ tuples, such that $B_i[t]$ satisfies properties 1, 2, and 3 described earlier for AAD. While these properties were proved in [1] for scalar states, the correctness of the properties also holds when $\mathbf{v}_i$ is a vector.

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2. In the t-th round, after obtaining set $B_i[t]$, process p_i computes its new state $\mathbf{v}_i[t]$ as follows. Form a multiset Z_i using the steps below:

   - Initialize Z_i as empty.
   - For each $C \subseteq B_i[t]$ such that $|C| = n - f \geq (d+1)f+1$, add to Z_i one deterministically chosen point from $\Gamma(\Phi(C))$. Since $|\Phi(C)| = |C| \geq (d+1)f+1$, by Lemma 1, $\Gamma(\Phi(C))$ is non-empty.

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   - For each $C \subseteq B_i[t]$ such that $|C| = n-f \geq (d+1)f+1$, add to Z_i one deterministically chosen point from $\Gamma(\Phi(C))$. Since $|\Phi(C)| = |C| \geq (d+1)f+1$, by Lemma 1, $\Gamma(\Phi(C))$ is non-empty.

   Note that $|Z_i| = \binom{|B_i[t]|}{n-f} \leq \binom{n}{n-f}$. Calculate

$$\mathbf{v}_i[t] = \frac{\sum_{\mathbf{z} \in Z_i} \mathbf{z}}{|Z_i|} \tag{9}$$

3. Each non-faulty process terminates after $1 + \lceil \log_{1/(1-\gamma)} \frac{U-\nu}{\epsilon} \rceil$ rounds, where γ $(0 < \gamma < 1)$ is a constant defined later in (11). Recall that ϵ is the parameter of the ϵ-agreement condition.

Note: $\Phi(B) = \{w_k : (p_k, w_k, t) \in B\}$

Proof of the correctness

Without loss of generality, suppose that m processes, $p_1, p_2, ..., p_m$ are non-faulty, where $m \geq n - f$, and the remaining $n - m$ processes are faulty.

Definition 1: A point $\mathbf{r}$ is said to be valid if there exists a representation of $\mathbf{r}$ as a convex combination of $v_k[t-1]$, $1 \leq k \leq m$. That is, there exists constants β_k, such that $0 \leq \beta_k \leq 1$ and $\sum_{1 \leq k \leq m} \beta_k = 1$, and

$$r = \sum_{1 \leq k \leq m} \beta_k v_k[t-1]$$

.

In general, there may exits multiple such convex combination representations of a valid point r. Moreover, it's obvious that at least one of the weights in any such convex combination must be $\geq \frac{1}{m} \geq \frac{1}{n}$.

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1. For any non-faulty process p_i, consider any $C \subseteq B_i[t]$, such that $|C| = n - f$. Because $n \geq (d+2)f + 1$, and thus $|\Phi(C)| = |C| = n - f \geq (d+1)f + 1$, by Lemma 1, $\Gamma(\Phi(C)) \neq \emptyset$. Therefore, Z_i will contain a point from $\Gamma(\Phi(C))$ for each C.

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   Because there are at most f faulty processes. Then there exists at least one $(n-2f)$-size subset of $\Phi(C)$ must be a subset of $\{v_1[t-1], v_2[t-1], ..., v_m[t-1]\}$. Therefore, all points in $\Gamma(\Phi(C))$ must be valid, and thus all the points in Z_i computed in *Step* 2 must be valid.

Proof of the correctness

In the following, we will break the proof into several parts.

1. All the points in Z_i computed in *Step* 2 must be valid.
2. Because $|B_i[t] \cap B_j[t] \geq n - f$. Therefore, there exists a set $C_{ij} \subseteq B_i \cap B_j$ such that $|C_{ij}| = n - f$. Therefore, Z_i and Z_j both contain one **identical** point from $\Gamma(\Phi(C_{ij}))$. Suppose the point is z_{ij}, as shown in Part 1, z_{ij} must be valid. Therefore, there must exists a non-faulty process, say $p_{g(i,j,t)}$, such that the weight associated with $v_{g(i,j,t)}[t-1]$ in the convex combination for z_{ij} is $\geq \frac{1}{m} \geq \frac{1}{n}$.

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3. Because $v_i[t]$ is computed as the average of the points in Z_i, and $|Z_i| = \binom{|B_i|}{n-f} \leq \binom{n}{n-f}$. Therefore, the weight of $v_{g(i,j,t)}[t-1]$ in $v_i[t]$ is $\geq \frac{1}{n\binom{n}{n-f}}$. Define

$$\gamma = \frac{1}{n\binom{n}{n-f}}$$

Proof of the correctness

Because we have proved that, at time step t, every z_k in Z_i is valid, and z_k can be treated as a convex combination of non-faulty processes' state vectors. Therefore, $v_i[t] = \frac{\sum_{z \in Z_i} z}{|Z_i|}$ can also be represented as a convex combination of non-faulty processes' state vectors (i.e. $\{v_1[t-1], ..., v_m[t-1]\}$).

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Therefore, we can get that for any non-faulty process's $v_i[t]$ is a convex combination of $\{v_1[0], v_2[0], ..., v_m[0]\}$, implying that the proposed algorithm satisfies the **validity** condition for approximate consensus. (note, $v_k[0]$ is the process p_k's input vector.)

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Let $v_{il}[t]$ denote the $l-th$ element of the vector of the state $v_i[t]$ of process p_i. Define $\Omega_l[t] = max_{1 \leq k \leq m} v_{kl}[t]$, the maximum value of l-th element of the vector state of non-faulty processes. Similiarily, $\mu_l[t] = min_{1 \leq k \leq m} v_{kl}[t]$. We have that:

$$\Omega_l[t] - \mu_l[t] \leq (1-\gamma)(\Omega_l[t-1] - \mu_l[t-1])$$

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Then we can repeat until $(1-\gamma)^t(\Omega_l[0] - \mu_l[0]) < \epsilon$, thus we can get

$$t > log_{1/(1-\gamma)} \frac{\Omega_l[0] - \mu_l[0]}{\epsilon}$$

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Simply assume that U is the upper bound of the input value, and v is the lower bound of the input value. We can eventually get that for each non-faulty process, it will terminate after $1 + log_{1/(1-\gamma)} \frac{U-v}{\epsilon}$, and $\epsilon-$agreement is ensured.

Proof of the correctness

In conclusion, the proposed algorithm can terminate within finite steps and the validity and the ϵ-agreement are both satisfied. Therefore, $n \geq (d+2)f + 1$ is sufficient for approximate consensus in asynchronous systems.

Conclusion

1. For a synchronous system, $n \geq max(3f+1, (d+1)f+1)$ is necessary and sufficient for achieving Byzantine vector consensus.
2. For an asynchronous system, $n \geq (d+2)f+1$ is necessary and sufficient to achieve *approximate* BVC.

QA

Thanks! QA?